Crystallography and the structure of $Z$-related sets

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# 1 Introduction

**The interval and Patterson functions** For a given chord $P$, the *interval function* measures the number of ways that the interval $x$ can be spanned between members of $P$:

$$IFUNC_x(P) = |\{a - b \equiv x, a, b \in P\}|.$$

Example: $P = \{0, 1, 4, 6\}$ in $\mathbb{Z}_{12}$. There are four ways to move by 0 (unison) and 1 way to move by each of the other intervals in 12-tone equal temperament (12-tet). Writing the interval function as a vector where the $i$th entry (beginning with $i = 0$) is $IFUNC_i(P)$, we have

$$IFUNC(P) = [411111111111].$$

Mathematicians call this the *Patterson function*. One of the underlying assumptions of the interval function is that a chord's "quality" is in large part a result of the quality of its constituent intervals.

**$Z$-related chords and homometric sets** If two chords have the same interval function, musicians say that they are $Z$-related.

Example: $P = \{0, 1, 4, 6\}$ and $Q = \{0, 1, 3, 7\}$ are $Z$-related, since their interval functions are identical.

Mathematicians say that these sets are *homometric*. Homometric metric sets were first studied in the 1940s by Bullough and others in relation to crystallography.

# 2 The Z-relation problem

**Necessary conditions** An important question is what are the necessary conditions for a set to possess a non-trivial $Z$-related/homometric partner. This is referred as the $Z$-relation problem.

Example: Let $P = \{0, \sqrt{2}, \sqrt{2} + 3, 6\}$. Is $P$ non-trivially $Z$-related to any other set? If so, which one(s)?[1]

**Understanding of $Z$-relations in Music Theory**

[1]Except for Case 2 in §3 below, the unit of distance is always the semitone. Thus, $\sqrt{2} \approx 1.41$ is the pitch-class approximately 1.41 semitones above pc 0.

1. Sets related by transposition and/or inversion are *trivially* $Z$-related.
2. If $P$ and $Q$ in $\mathbb{Z}_n$ are $Z$-related, then their complements with respect to $\mathbb{Z}_n$, $\bar{P}$ and $\bar{Q}$, are also $Z$-related.
3. In $\mathbb{Z}_{2n}$, $n$-note sets are $Z$-related to their complements.
4. Non-trivial $Z$-relations occur for sets of 4 or more notes.
5. Non-trivial $Z$-relations do not occur for equal tempered systems of fewer than 8 pitch-classes.
6. Soderberg's "Q-inversion" preserves interval content.

## 3 Some cases of $Z$-relations

**Case 1** (Bullough, Rosenblatt)

$$\{0, x, x + 3, 6\} \, Z \, \{0, x + 3, 6, x + 6\}$$

Examples:

For $x = 1$, $\{0, 1, 4, 6\}$ (4-Z15) $Z$ $\{0, 4, 6, 7\}$ (4-Z29).

For $x = \sqrt{2}$, $\{0, \sqrt{2}, \sqrt{2} + 3, 6\} \, Z \, \{0, \sqrt{2} + 3, 6, \sqrt{2} + 6\}$.

Noting that the second chord in Case 1 is inversionally related to $\{-x, 0, -x + 3, 6\}$ leads to an alternate formulation:

$$\{0, x, x + 3, 6\} \, Z \, \{-x, 0, -x + 3, 6\}.$$

These two different formulations of the same case can be extended in a separate manner. (See cases 3 and 4 below.)

Case 1 generates all possible $Z$-related pairs of 4 notes, except the following case. (This was proved by Rosenblatt.)

**Case 2** In $\mathbb{Z}_{13}$,

$$\{0, 1, 4, 6\} \, Z \, \{0, 2, 3, 7\}.$$

**Case 3** We can extend the alternate formulation of Case 1 in the following manner. Replace the minor third by two inversionally symmetric sets where the axes of symmetry are a minor third apart.

Let $P = \{0, 6\}$, $R_1 = \mathbf{I}_0(R_1)$, $R_2 = \mathbf{I}_6(R_2)$, and $S_x = P \cup T_x(R_1 \cup R_2)$. Then $S_x$ and $S_{-x}$ are $Z$-related.

By varying the choices of $R_1$, $R_2$, and $x$, case 4 generates 12 of the 19 $Z$-related pairs of 6 or fewer notes in 12-tet. (See Table 1.) There are a total of 85 $Z$-related pairs in 8-tet through 14-tet, not including those pairs where the cardinality is greater than half of the number of pcs. Case 2 generates 50 of these pairs, or 59%. (Of course one of the remaining pairs corresponds to case 3.)

Case 3 can be realized using Soderberg's Q-inversion, but to do so one must understand the basic structure of these sets. In other words, one must understand Case 3 to find the appropriate Q-inversion.

| $R_1$ | $R_2$ | $x$ | $S_x$ | $S_{-x}$ |
|---|---|---|---|---|
| $\{0\}$ | $\{3\}$ | 1 | $\{0,1,4,6\} \in$ 4-Z15 | $\{11,0,2,6\} \in$ 4-Z29 |
| $\{0\}$ | $\{2,4\}$ | 1 | $\{0,1,3,5,6\} \in$ 5-Z12 | $\{11,0,1,3,6\} \in$ 5−Z-36 |
| $\{-2,2\}$ | $\{3\}$ | 1 | $\{11,0,3,4,6\} \in$ 5-Z18 | $\{9,0,1,2,6\} \in$ 5-Z38 |
| $\{-1.5,1.5\}$ | $\{2.5,3.5\}$ | 0.5 | $\{10,0,1,2,3,6\} \in$ 6-Z39 | $\{11,0,2,3,4,6\} \in$ 6-Z10 |
| $\{-2.5,2.5\}$ | $\{1.5,4.5\}$ | 0.5 | $\{9,0,1,2,4,6\} \in$ 6-Z46 | $\{10,0,2,3,5,6\} \in$ 6-Z24 |
| $\{-2.5,2.5\}$ | $\{0.5,5.5\}$ | 1.5 | $\{8,11,0,1,4,6\} \in$ 6-Z48 | $\{11,0,2,4,6,7\} \in$ 6-Z26 |
| $\{-1,0,1\}$ | $\{3\}$ | 2 | $\{0,1,2,3,5,6\} \in$ 6-Z3 | $\{9,10,11,0,1,6\} \in$ 6-Z36 |
| $\{-0.5,0.5\}$ | $\{2.5,3.5\}$ | 1.5 | $\{0,1,2,4,5,6\} \in$ 6-Z4 | $\{10,11,0,1,2,6\} \in$ 6-Z37 |
| $\{-2,0,2\}$ | $\{3\}$ | 1 | $\{9,11,0,1,2,6\} \in$ 6-Z40 | $\{11,0,1,3,4,6\} \in$ 6-Z11 |
| $\{-1.5,1.5\}$ | $\{2.5,3.5\}$ | 0.5 | $\{11,0,2,3,4,6\} \in$ 6-Z10 | $\{10,0,1,2,3,6\} \in$ 6-Z39 |
| $\{-4,0,4\}$ | $\{3\}$ | 1 | $\{9,0,1,4,5,6\} \in$ 6-Z44 | $\{7,11,0,2,3,6\} \in$ 6-Z19 |
| $\{-1,1,6\}$ | $\{3\}$ | 2 | $\{0,1,3,5,6\} \in$ 6-Z25 | $\{9,11,0,1,4,6\} \in$ 6-Z47 |

Table 1: The Z-related pcsets $S_x$ and $S_{-x}$ in 12-tet, where $S_x = \{0,6\} \cup \mathbf{T}_x(R_1 \cup R_2)$.

**Case 4a** (Bullough) We can extend Case 1 in the following manner. Let $P$ be the $m$-fold division of the octave that contains $\{0\}$, $P = \left\{\frac{12j}{m}\right\}_{j=0}^{m-1}$. Divide the smallest interval of $P$ into $n$ parts, $Q = \left\{\frac{12k}{mn}\right\}_{k=0}^{n-1}$, and transpose this by an arbitrary interval, $\mathbf{T}_x(Q)$. Then

$$P \cup \mathbf{T}_x(Q)\ Z\ P \cup \mathbf{T}_{x+\frac{12k}{mn}}(Q).$$

**Case 4b** (Bullough) This is the same as Case 4a with the exception that members of $Q$ are not necessarily equal to $\frac{12k}{mn}$. Rather, each member of $Q$, $q_k$, belongs to the $m$-fold division of the octave that *contains* $\frac{12k}{mn}$. That is, $Q = \{q_k\}_{k=0}^{n-1}$, where $q_k \equiv \frac{12k}{mn} \bmod \frac{12}{m}$. With this modification, the result in Case 4a also obtains.

# 4 Mathematical approaches to homometric sets

**The Fourier transform** Quinn (2007) uses the Fourier transform as part of an extensive theory of chord quality. In continuous pc space, the Fourier transform of each pcset gives rise to an infinite series of Fourier components. Intuitively, the $k^{th}$ Fourier component of a pcset $P$ is given by the following steps:

1. begin with a 12-pc clockface;
2. for each $p$ in $P$ draw an arrow from the center of the clock to $kp$'oclock (even if this gives rise to non-integer values);
3. add the arrows together by placing, in any order, the tail of one to the head of another; and
4. the arrow extending from the center of the clock to the head of the last arrow in the summation is the $k^{th}$ Fourier component.[2]

The length of this arrow is the magnitude of this component, written $|\hat{P}(k)|$. The importance of the Fourier transform for the Z-relation is that if $|\hat{P}(k)| = |\hat{Q}(k)|$ for all $k$, then $P\ Z\ Q$.

[2]Formally, the Fourier transform of pcset $P$ (more precisely, the characteristic function of $P$) is $\sum_{p \in P} e^{-2\pi i p z}$.

**The $Z$-relation as an algebraic property** Rosenblatt and others represent sets in $\mathbb{R}$ or $\mathbb{R}/\mathbb{Z}$ as polynomials. For our purposes, the pitch-class set $P$ can be represented by the polynomial $P(y) = \sum_{p \in P} y^p$, where $y^{12} = y^0 = 1$. For example, $Q = \{0, 1, 4, 6\}$ is represented by $Q(y) = y^0 + y^1 + y^4 + y^6$. For any two pcsets represented as polynomials, $D(y)$ and $E(y)$, there is a transformation, $U(y)$, that takes us from $D$ to $E$:

$$D(y)U(y) = E(y), \text{ or } U(y) = \frac{E(y)}{D(y)}.$$

For example, transposition by $x$ is given by the transformation $U(x) = y^x$; e.g.,

$$Q(y) \cdot y^2 = (y^0 + y^1 + y^4 + y^6)y^2 = y^2 + y^3 + y^6 + y^8 = T_2(Q)(y).$$

$U$ is called a *spectral unit* if $U(y)U^*(y) = y^0 = 1$, where $U^*$ is the complex conjugate of $U$. For example, transposition by $x$ is a spectral unit: $y^x y^{-x} = y^0 = 1$.

**Theorem (from Rosenblatt)** *If $D, E \in \mathbb{R}/\mathbb{Z}$ or $\mathbb{R}$, then $D$ and $E$ are homometric (Z-related) if and only if there exists a spectral unit $U$ such that $D(x)U(x) = E(x)$.*

Thus, the existence of homometric sets relates to the factorization of polynomials. Whether this algebraic property has a relevant musical interpretation depends on the particular case.

## 5 Bibliography


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